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Self-organized criticality in a stochastic spring-block model

Y. N. Lu
Institute of Low Energy Nuclear Physics, Beijing Normal University, Beijing 100875, China

E. J. Ding
Chinese Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing, 100080, China;
Institute of Low Energy Nuclear Physics, Beijing Normal University, Beijing 100875, China;
and Institute of Theoretical Physics, Academia Sinica, Beijing 100080, China
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A stochastic spring-block model with both global correlation and local interaction is considered in terms of the cellular automaton. It is shown that there is a scaling relation $D(\Delta)\propto\Delta^{\xi}\exp(-\Delta/\Delta_0)$ between the slip size Δ and its probability $D(\Delta)$ with a universal exponent $\xi=-1.5$. The value of Δ_0 is nonuniversal. The behavior of this model is surprisingly close to the Gutenberg-Richter law and that of the recent experiment [H. J. S. Feder and J. Feder, Phys. Rev. Lett. **66**, 2669 (1991)].

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The spring-block model was originally considered by Burridge and Knopoff to be a realistic model of earthquakes [1]. Based on statistics for earthquakes, Gutenberg and Richter [2] in 1954 gave a scaling law, according to which the number $d\Omega$ of observed earthquakes with released energy between E and $E+dE$ was

$$d\Omega\propto E^{-B}dE \ . \tag{1}$$

A thorough reexamination of earthquake catalogs has recently concluded that small earthquakes are distributed with $B\simeq1.5-1.6$, whereas large earthquakes of magnitude 7 or above have a large $B\simeq2$ [3,4]. Since Bak, Tang, and Wiesenfeld introduced the concept of self-organized criticality [5], the spring-block models have also served as a paradigm for this concept [6–8]. Recently some authors [9] studied experimentally a stick-slip process of dragging sandpaper across a carpet. In their experiment the reduction of force in a slip event had a power-law probability distribution, which was similar to the Gutenberg-Richter law (1) for earthquakes. One may find that the global correlation was introduced in their experimental setup, though it was not explicitly mentioned. However, to our knowledge, most of the spring-block models studied up to the present have been restricted to the case without global correlation. The role of global correlation has not yet been fully understood. In a recent work [10] we proposed a globally coupled string-block model where the local interaction was ignored. The probability $D(\Delta)$ of the slip with size Δ in that model could exactly be calculated, based on the approach in Ref. [11]. In this Rapid Communication we propose a general spring-block model that includes both the local interaction and the global correlation. Self-organized criticality appears in this model. The distinctive behavior in this model is that the probability distribution of the slip size conforms quite closely to the Gutenberg-Richter law.

Consider a total number N of blocks frictionally contacting a fixed carpet (see Fig. 1). A rigid bar is connected to each block by individual springs. These springs are assumed to have the same elastic modulus κ_2. The bar is also connected to a trunk spring with the elastic modulus κ_1. This trunk spring corresponds to the nylon fishing line in the experimental setup of Ref. [9]. The tip A of the trunk spring is drawn very slowly. In addition, another set of springs with the elastic modulus κ_3 connects each block to its two nearest neighbors. If every block sticks to the carpet the force exerted on the bar is

$$F=\kappa_1(r-z)=\kappa_2\sum_{i=1}^{N}(z-x_i) \ , \tag{2}$$

where r, z, and x_i are the positions of tip A, the bar, and the ith block, respectively, as they are calculated from properly chosen origins. The initial values of x_i's may be different from one another. The static friction of the ith block is

$$f_i=\kappa_2(z-x_i)+\kappa_3(x_{i-1}+x_{i+1}-2x_i) \ . \tag{3}$$

In this paper the periodic boundary condition $x_0=x_N$ is assumed, so every block has two nearest neighbors. It is quite clear that $F=\sum_{i=1}^{N}f_i$. As we steadily draw tip A, the force F and r, z, and all f_i's increase with time uniformly until one block, say the jth one, experiences a force f_j, which reaches the maximum friction f_t. Then this block will move to a new position, say x_j^*, and stick there. The static friction between the jth block and the

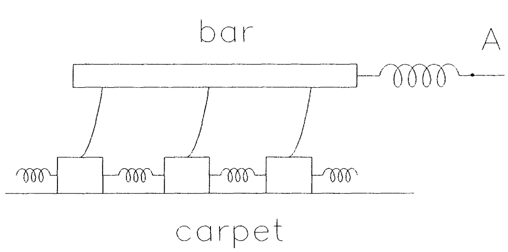

FIG. 1. Sketch of the spring-block system under consideration.

carpet at the new position is assumed to be $f_j^* \in (0, f_t)$ before the bar moves. We know from Eq. (3) that $x_j^* = x_j + (f_j - f_j^*)/(\kappa_2 + 2\kappa_3)$. The bar then moves a little (say dz), though tip A does not move. We have from Eq. (2) that

$$dz = \frac{\kappa_2(x_j^* - x_j)}{\kappa_1 + \kappa_2 N} , \tag{4}$$

and from Eq. (3) that the value of f_j also changes to $f_j' = f_j^* + \kappa_2 dz$. By using Eq. (4) we have that $\kappa_2 dz = \alpha u/(1+2\beta)N$; here $\alpha = 1/(1+\kappa_1/\kappa_2 N)$, $\beta = \kappa_3/\kappa_2$, and

$$u = f_j - f_j^* . \tag{5}$$

By properly choosing the unit of length, the maximum friction between each block and the carpet is taken to be unity, i.e., $f_t = 1$. Therefore we get the rules for the cellular automaton

$$f_j \to f_j^* + \frac{\alpha}{(1+2\beta)N} u , \quad f_j \geq 1$$

$$f_i \to \begin{cases} f_i + \dfrac{\alpha + N\beta}{(1+2\beta)N} u , & i = j \pm 1 \\[2ex] f_i + \dfrac{\alpha}{(1+2\beta)N} u & \text{otherwise} . \end{cases} \tag{6}$$

The value of f_j^* is chosen randomly in the interval [0,1] through a given function $p_0(f)$, which is called the prestrain distribution [10]. If, after the rearrangements, there are still some blocks with static friction larger than the threshold, the rules (6) must be used repeatedly until all blocks do not move anymore. Before tip A of the trunk spring is drawn further, the total number Δ of blocks that are simulataneously moved in a chain reaction is called the size of the slip. Notice that the total force on the bar F may decrease after the automaton rules are used. The dissipation is proportional to

$$R = \frac{1-\alpha}{1+2\beta} . \tag{7}$$

From the rules (6) we see that when a block slips the static friction on every other block increases. The increment $u\alpha/(1+2\beta)N$ corresponds to the global correlation, while the additional increment $u\beta/(1+2\beta)$ of two nearest neighbors is due to the local interaction. As long as $\alpha > 0$ the system is globally correlated. As $\kappa_1 \to \infty$ the model becomes the one-dimensional version of that in Ref. [8], with stochastic prestrain and a different boundary condition. When $\kappa_3 = 0$, i.e., $\beta = 0$, it returns to the exactly solvable case discussed in Ref. [10].

Here we have at least three different scales of time: the time τ_A needed for tip A to move a unit distance, the relaxation time τ_R for the redistribution of the load among the springs, and the time τ_S needed for the moving block to re-adhere to the carpet. Since the time interval between earthquakes is much larger than the actual duration of an earthquake, a common feature of the theoretical spring-block models is that both τ_R and τ_S are much smaller than τ_A. In the present discussion we consider the case of $\tau_S \ll \tau_R$, i.e., the moving block will re-adhere to a new place before the springs rearrange their strain distribution. The alternative case $\tau_S \gg \tau_R$ will be discussed elsewhere.

Let us consider the scaling behavior of the distribution $D(\Delta)$. When $\beta = 0$ we have the asymptotic expression [10]

$$D(\Delta) = C\Delta^{\xi} \exp(-\Delta/\Delta_0) \quad (\Delta \gg 1) , \tag{8}$$

where $C = \exp(\theta s) \int_0^1 p_0(r) \exp(-sr) dr / \sqrt{2\pi} \theta s \equiv C_0$, $\xi = -1.5$, and $\Delta_0 = 1/(\theta s - 1 - \ln\theta s)$, with $s = \alpha / \int_0^1 (1-r) p_0(r) dr$, and $\theta \in [0, 1]$. Since Eq. (8) is correct only for large Δ's this formula might not be normalized. The value of θs is approximately equal to α for α close to 1, so we have that $\Delta_0 \simeq 1/(\alpha - 1 - \ln\alpha) \equiv G_\alpha$. When $\beta > 0$ the numerical method has to be used. The simulations show that the distribution $D(\Delta)$ for large Δ's can still be expressed in the form of Eq. (8). For given values of parameters the data obtained numerically can be fitted by

$$\ln[D(\Delta)\Delta^{-\xi}] = -\Delta/\Delta_0 + \ln C \quad (\Delta \gg 1) . \tag{9}$$

For a given value of ξ the standard least-squares fit gives the corresponding value of Δ_0. The value of ξ is chosen in such a way that the minimum deviation from a linear dependence between $\ln[D(\Delta)\Delta^{-\xi}]$ and Δ is obtained. It is found that within a numerical accuracy of about ± 0.2, the value $\xi = -1.5$ is universal for all parameter values. The value of Δ_0, however, depends on the parameters α and β. The interesting thing we found in the numerical simulation is that the cutoff size Δ_0 can be factorized as

$$\Delta_0 \simeq G_\beta G_\alpha . \tag{10}$$

The factor G_α depends only on $\alpha = 1/(1+\kappa_1/\kappa_2 N)$, as shown above, and G_β on the parameter $\beta = \kappa_3/\kappa_2$. Here the factor G_β can be approximately expressed as $G_\beta \simeq 1 + \nu\beta$, with $\nu \simeq 3$ for $\beta \leq 10$. Hence Eq. (10) turns out asymptotically to be

$$\Delta_0 \simeq (1+3\beta)/(\alpha - 1 - \ln\alpha) . \tag{11}$$

The numerical results confirm the independent scaling property for $\alpha \geq 0.3$ and $\beta \leq 10$, as shown in Fig. 2. However, when $\alpha = 0.1$ the results are no longer regarded as fairly satisfactory, at least for small β.

When $\beta > 0$ we find that the coefficient $C \simeq C_0$ in Eq. (8). With the help of Eqs. (8) and (10) the distribution function $D(\Delta)$ for a variety of parameter values could, after proper rescaling, be fitted in an universal function as

$$\frac{D(\Delta)}{D_0} = \Phi \left[\frac{\Delta}{\Delta_0} \right] , \tag{12}$$

where $\Phi(X) = X^{-3/2} \exp(-X)$, and $D_0 = C_0 G_\alpha^{-3/2} G_\beta^{-3/2}$. The function $\Phi(X)$ is universal, dependent neither on the prestrain distribution $p_0(f)$ nor the parameters α and β. The coefficient D_0, however, is nonuniversal. Note that this result is corrected only for a large size of slip. The numerical results are shown in Fig. 3. In order to show the system size N dependence explicitly, we can obtain from Eq. (10) that for $N \gg 1$

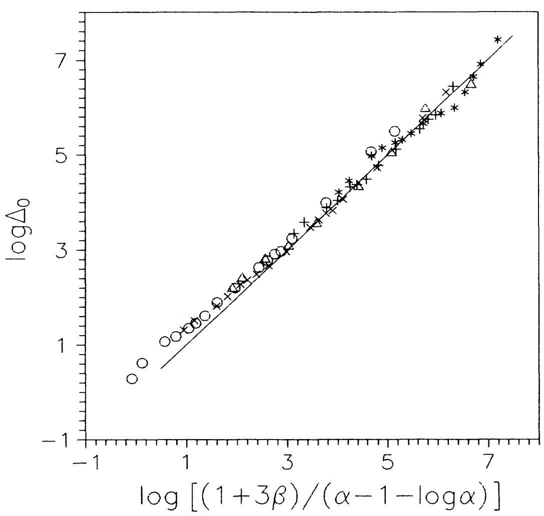

FIG. 2. Dependence of Δ_0 on the values of $(1+3\beta)/(\alpha-1-\ln\alpha)$. The straight line indicates the bisector. $\circ$, $\alpha=0.1$; $\times$, $\alpha=0.3$; $\triangle$, $\alpha=0.5$; $+$, $\alpha=0.7$; and $*$, $\alpha=0.8$.

$$\Delta_0 \simeq \frac{2\kappa_2(\kappa_2+3\kappa_3)}{\kappa_1^2} N \left[N - \frac{2\kappa_1}{3\kappa_2} \right] . \tag{13}$$

For large N with κ_1, κ_2, and κ_3 given we have $\Delta_0 \propto N^2$. We conclude that the system displays self-organized criticality, because for any values of the parameters κ_1, κ_2, and κ_3 the cutoff size of the slip, Δ_0, tends to infinity as long as $N \rightarrow \infty$; and with finite system $N \gg \kappa_1/\kappa_2$ the distribution of the slip size $D(\Delta)$ could be fitted by the so-called finite-size scaling method [12] as

$$\frac{D(\Delta)}{N^{-3}} = \phi \left[\frac{\Delta}{N^2} \right] . \tag{14}$$

It is easy to find from Eqs. (8) and (13) that $\phi(X) = C' X^{-3/2} \exp(-X/X_0)$, with $X_0 = 2\kappa_2(\kappa_2+3\kappa_3)/\kappa_1^2$ and C' a constant, depending on κ_1, κ_2, and κ_3. Moreover, it is clear that the averaged slip size $\langle \Delta \rangle$ is

$$\langle \Delta \rangle = \sum_{\Delta=1}^{\infty} \Delta D(\Delta) \propto \Delta_0^{1/2} , \tag{15}$$

and that the variation of the slip size is

$$(\langle \Delta^2 \rangle - \langle \Delta \rangle^2)^{1/2} \propto \Delta_0^{3/2} . \tag{16}$$

The relations (15) and (16) are also verified by numerical experiments.

It is quite inconvenient to count the number of slipping blocks in experiments. One could instead measure the force on the bar, as was done in the experiment of Ref. [9]. The force per block on the bar

$$f = \frac{1}{N} \sum_{i=1}^{N} f_i$$

varies with time t. The value of f falls rapidly in a slip, and increases steadily between two slips. Numerical simulations show that the fall δf of f is proportional to the size of slip and that its distribution function is in agreement in form with that of $D(\Delta)$, see Fig. 4. The distribution $P(\delta f)$ is qualitatively the same as that in the experiment on sandpaper in Ref. [9].

The infinite correlation length is a characteristic of the

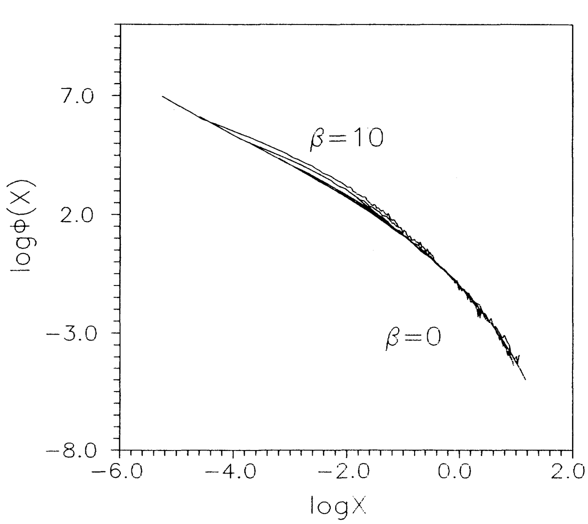

FIG. 3. The slip-size distribution $D(\Delta)$ can be fitted by universal scaling function $\Phi(X)$. The numerical results for β=0.0, 0.5, 1.0, 2.0, 5.0, and 10.0 are presented, where α=0.5.

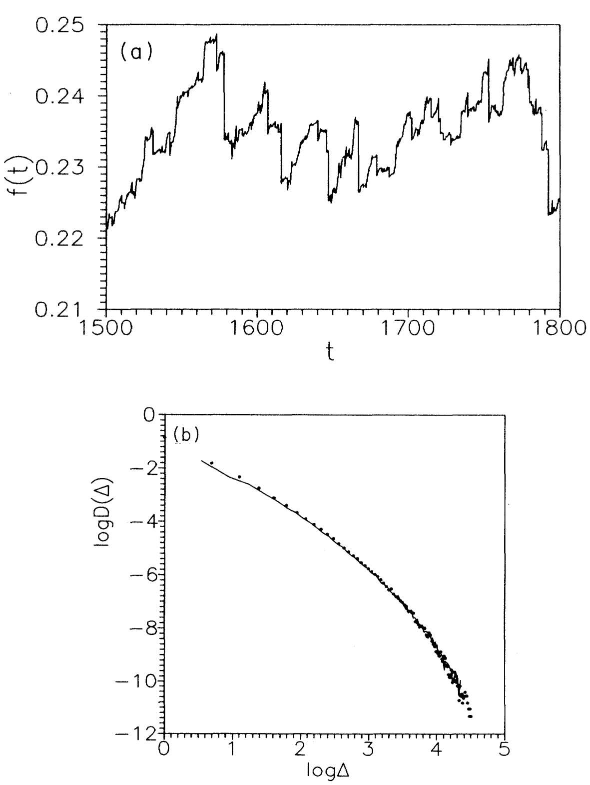

FIG. 4. (a) Force per block f vs time t. (b) The solid line represents the distribution $P(\delta f)$ of the fall δf of f in the slips. In a suitable scale it is in agreement with $D(\Delta)$, indicated by dots. Here α=0.5, β=1.0, and N=500. Notice that the results are qualitatively the same as those in Ref. [9].

criticality. We define the correlation function as

$$c(r)=\frac{1}{T}\sum_{t=1}^{T}c_t(r)\ , \tag{17}$$

where

$$c_t(r)=\frac{1}{N}\sum_{i=1}^{N}\delta x_i \delta x_{i+r}\ ,$$

with δx_i the displacement of the ith block in the tth slip event. The numerical calculation gives the results shown in Fig. 5. The correlation function decays exponentially,

$$c(r)\propto \exp(-r/\eta) \quad \text{when } r<r_c\ ,$$

while the values of η and r_c depend on the parameters. When $r>r_c$ the correlation function becomes a constant. The finite value of r_c is explained by the fact that the system is globally correlated when $\alpha>0$. The value of r_c tends to infinity as $\alpha\rightarrow 0$. η is called the correlation length, which increases as α increases. A common feature of the criticality is that the correlation length goes to infinity. In the present model it occurs when $\Delta_0\rightarrow\infty$.

It should be mentioned that in the case of conservation ($\kappa_1=0$, for instance), the system cannot stay in the stick-slip state. In fact, the system would at last evolve into a state where every block moves simultaneously. However, as long as $\kappa_1>0$, $\kappa_2>0$, and $\kappa_3<\infty$, we always have $R>0$ for a finite-size system, and the system stays in the stick-slip state. Alternatively, if we introduce the dissipation of the force by taking

$$u=f_t-f_j^*\ , \tag{18}$$

instead of Eq. (5), the system will stay in the critical state for any choice of parameters.

In the above simulation we have used the uniform prestrain distribution $p_0(f)=1$. We have also simulated other choices of prestrain distribution $p_0(f)$. The results show that the exponent $\xi=-1.5$ in Eq. (8) and the function $\Phi(X)$ are universal, dependent neither on the prestrain distribution $p_0(f)$ nor on the parameters α and β. The values of D_0 and Δ_0 are nonuniversal. The result obtained in the present Rapid Communication is extremely close to the Gutenberg-Richter law if we realize that the energy released in a slip event is proportional to the fall of the force $N\delta f$. The simulation on a two-dimensional spring-block system shows the results, which are qualitatively the same as on the one-dimensional system here, and they will be presented elsewhere.

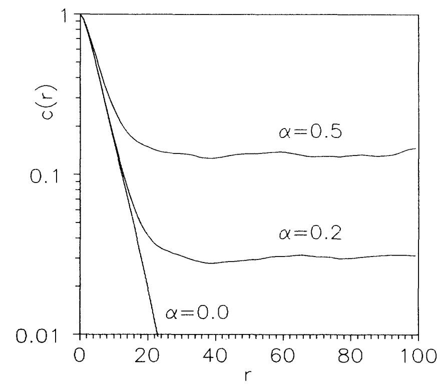

FIG. 5. Correlation functions $c(r)$. The calculations are carried out for $\beta=10$ and $N=200$.

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